

COPRIME INVARIABLE GENERATION AND MINIMAL-EXPONENT GROUPS

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ABSTRACT. A finite group G is *coprimely-invariably generated* if there exists a set of generators $\{g_1, \dots, g_u\}$ of G with the property that the orders $|g_1|, \dots, |g_u|$ are pairwise coprime and that for all $x_1, \dots, x_u \in G$ the set $\{g_1^{x_1}, \dots, g_u^{x_u}\}$ generates G .

We show that if G is coprimely-invariably generated, then G can be generated with three elements, or two if G is soluble, and that G has zero presentation rank. As a corollary, we show that if G is any finite group such that no proper subgroup has the same exponent as G , then G has zero presentation rank. Furthermore, we show that every finite simple group is coprimely-invariably generated.

Along the way, we show that for each finite simple group S , and for each partition π_1, \dots, π_u of the primes dividing $|S|$, the product of the number $k_{\pi_i}(S)$ of conjugacy classes of π_i -elements satisfies

$$\prod_{i=1}^u k_{\pi_i}(S) \leq \frac{|S|}{2|\text{Out } S|}.$$

1. INTRODUCTION

Following [10] and [14], we say that a subset $\{g_1, \dots, g_u\}$ of a finite group G *invariably generates* G if $\{g_1^{x_1}, \dots, g_u^{x_u}\}$ generates G for every choice of $x_i \in G$.

Definition 1.1. A finite group G is *coprimely invariably generated* if there exists a set of invariable generators $\{g_1, \dots, g_u\}$ of G with the property that the orders $|g_1|, \dots, |g_u|$ are pairwise coprime.

Our main result says that a coprimely invariably generated group can be generated with very few elements. Let $d(G)$ denote the minimal number of generators of G .

Theorem 1.2. *Let G be a coprimely invariably generated group. Then $d(G) \leq 3$.*

Notice that coprime invariable generation is the combination of two properties: the existence of an invariable generating set and the existence of a set of generators of coprime orders. It is worth noticing that neither of these properties suffices to obtain an upper bound on the smallest cardinality of generators of a finite group G . Clearly any finite group G contains an invariable generating set (consider the set of representatives of each of the

conjugacy classes). Moreover for every $t \in \mathbb{N}$ there exists a finite (supersoluble) group G with the property that $d(G) = t$ and G can be generated with t elements of coprime order (see Proposition 3.2).

For general G , the bound on $d(G)$ given in Theorem 1.2 cannot be improved: there exists a coprimely invariably generated group G with $d(G) = 3$ (see Proposition 3.1). However better results hold under additional assumptions. For example, we have a stronger result for finite soluble groups.

Theorem 1.3. *Let G be a coprimely invariably generated group. If G is soluble, then $d(G) \leq 2$.*

A motivation for our interest in coprime invariable generation is the fact that this property is satisfied by finite groups without proper subgroups of the same exponent (we will call these groups *minimal exponent groups*). Indeed, assume that G is a minimal exponent group with $e := \exp(G) = p_1^{n_1} \cdots p_t^{n_t}$. Then for every i , the group G contains an element g_i of order $p_i^{n_i}$. Clearly $\exp\langle g_1^{x_1}, \dots, g_t^{x_t} \rangle = e$, for every $x_1, \dots, x_t \in G$. Hence our assumption that no proper subgroup of G has exponent e implies that $G = \langle g_1^{x_1}, \dots, g_t^{x_t} \rangle$, so G is coprimely invariably generated. In particular, as a corollary of Theorems 1.2 and 1.3, we deduce a result already proved in [18] and [8]: a finite group G contains a 3-generated subgroup H with $\exp(G) = \exp(H)$ and if G is soluble there exists indeed a 2-generated subgroup H of G with $\exp(G) = \exp(H)$.

Notice that the example given in Proposition 3.1 of a coprimely invariably generated group G which is not 2-generated is not minimal exponent. Indeed, the property of being minimal exponent is much stronger than coprime invariable generation.

Whereas the bound $d(G) \leq 3$ in Theorem 1.2 cannot be improved, we have no example of a finite minimal exponent group G which cannot be generated by 2 elements and the following interesting question is open: *is it true that any finite group G contains a 2-generated proper subgroup with the same exponent?* We think that the study of coprimely invariably generated groups could help to answer this question.

The minimal exponent property is not inherited by quotients; conversely, all epimorphic images of a coprimely invariably generated group (and consequently of a minimal exponent group) are coprimely invariably generated. From this point of view, studying coprimely invariably generated groups yields information about quotients of minimal exponent groups.

Another result in this paper concerns the presentation rank of coprimely invariably generated groups. The *presentation rank* $pr(G)$ of a finite group G is an invariant whose definition comes from the study of relation modules (see [4] for more details). Let I_G denote the augmentation ideal of $\mathbb{Z}G$, and $d(I_G)$ the minimal number of elements of I_G needed to generate I_G as a G -module, then $d(G) = d(I_G) + pr(G)$ [21]. It is known that $pr(G) = 0$ for

many groups G , including all soluble groups, all Frobenius groups and all 2-generated groups.

Theorem 1.4. *Let G be a coprimely invariably generated group. Then G has zero presentation rank.*

As an immediate corollary, we get the following.

Theorem 1.5. *Let G be a finite group such that no proper subgroup has the same exponent as G . Then G has zero presentation rank.*

As a further contribution to the understanding of coprimely invariably generated groups, we present the following theorem.

Theorem 1.6. *Let G be a finite simple group. Then G is coprimely invariably generated.*

Finally, the following result on conjugacy classes of finite simple groups may be of independent interest. If G is a group and $\pi = \{p_1, \dots, p_k\}$ a set of primes, then $|G|_\pi$ denotes the π -part of $|G|$ and an element of G whose order is $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, for some $\alpha_1, \dots, \alpha_k \in \mathbb{Z}_{\geq 0}$, is a π -element. Notice that the identity is a π -element. We let $k_\pi(G)$ denote the number of conjugacy classes of π -elements of G .

Theorem 1.7. *Let S be a finite simple group and let π_1, \dots, π_u be a partition of $\pi(S)$. Then*

$$\prod_{i=1}^u k_{\pi_i}(S) \leq \frac{|S|}{2|\text{Out } S|}.$$

This paper is structured as follows. In Section 2 we present some background information needed for our proofs. In Section 3 we construct two interesting examples, exploring the necessity and sufficiency of coprime invariable generation in controlling minimal generation and exponent. In Section 4 we prove Theorem 1.3, then in Section 5 we prove Theorems 1.2 and 1.4. Finally, in Sections 6 and 7 we prove Theorems 1.6 and 1.7, respectively.

2. BACKGROUND MATERIAL

In this section we introduce primitive monolithic groups and crown-based powers, and collect some information about their minimal number of generators, and about their presentation rank.

A group L is *primitive monolithic* if L has a unique minimal normal subgroup A , and trivial Frattini subgroup. We define the *crown-based power* of L of size t to be

$$L_t = \{(l_1, \dots, l_t) \in L^t \mid l_1 A = \cdots = l_t A\} = A^t \text{diag}(L^t).$$

In [4] it was proved that, given a finite group G , there exist a primitive monolithic group L and a positive integer t such the crown-based power L_t of size t is an epimorphic image of G and $d(G) = d(L_t) > d(L/\text{soc}(L))$.

The minimal number of generators of a crown-based power L_t in the case where A is abelian can be computed with the following formula:

Theorem 2.1. [6, Proposition 6] *Let L be a primitive monolithic group with abelian socle A , and let t be as above. Define*

$$r_L(A) = \dim_{\text{End}_{L/A}(A)} A \quad s_L(A) = \dim_{\text{End}_{L/A}(A)} H^1(L/A, A)$$

and set $\theta = 0$ if A is a trivial L/A -module, and $\theta = 1$ otherwise. Then

$$d(L_t) = \max \left(d(L/A), \theta + \left\lceil \frac{t + s_L(A)}{r_L(A)} \right\rceil \right)$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x .

A result of Aschbacher and Guralnick [1] assures us that $s_L(A) < r_L(A)$:

Theorem 2.2. [1] *Let p be a prime and G be a finite group. If A is a faithful irreducible G -module over \mathbb{F}_p , then $|H^1(G, A)| < |A|$.*

For soluble G we will use the following (the proof can be found in [22]):

Theorem 2.3 (Gaschütz). *Let p be a prime. If G is a finite p -soluble group and A is a faithful irreducible G -module over \mathbb{F}_p , then $|H^1(G, A)| = 0$.*

When A is non-abelian, $d(L_t)$ can be evaluated using the following, where $P_{L,A}(k)$ denotes the conditional probability that k randomly chosen elements of L generate L , given that they project onto generators for L/A .

Theorem 2.4. [4, Theorem 2.7] *Let L be a monolithic primitive group with non-abelian socle A , and let $d \geq d(L)$. Then $d(L_t) \leq d$ if and only if*

$$t \leq \frac{P_{L,A}(d)|A|^d}{|C_{\text{Aut } L}(L/A)|}.$$

Bounds on $P_{L,A}(d)$ were studied in [8] and [20], achieving the strong result:

Theorem 2.5. [8] *Let L be a primitive monolithic group with socle A . Then $P_{L,A}(d) \geq 1/2$.*

We finish this introductory section with a result on presentation rank.

Theorem 2.6. *Let G be a finite group and let L_t be a crown based power of a primitive monolithic group L such that L_t is a homomorphic image of G and $d(G) = d(L_t) > d(L/\text{soc}(L))$. If $\text{soc}(L)$ is abelian, then $\text{pr}(G) = 0$.*

Proof. For an irreducible G -module M , we set

$$r_G(M) = \dim_{\text{End}_G(M)} M \quad s_G(M) = \dim_{\text{End}_G(M)} H^1(G, M)$$

and define

$$h_G(M) = \theta + \left\lceil \frac{s_G(M)}{r_G(M)} \right\rceil$$

where $\theta = 0$ if M is a trivial and $\theta = 1$ otherwise.

Assume that $A = \text{soc}(L)$ is abelian. Let $\delta_G(A)$ be the largest integer k such that the crown based power L_k is a homomorphic image of G , and note that

$$d(L_{\delta_G(A)}) = d(L_t) = d(G).$$

By [7, Proposition 9], the integer $\delta_G(A)$ is the number of complemented chief factors G -isomorphic to A in any chief series of G . Since

$$r_G(A) = r_L(A)$$

and

$$s_G(A) = \dim_{\text{End}_G(A)} H^1(G, A) = \delta_G(A) + \dim_{\text{End}_{L/A}(A)} H^1(L/A, A)$$

(see e.g. [1.2] in [16]), it follows that

$$h_G(A) = \theta + \left\lceil \frac{\delta_G(A) + s_L(A)}{r_G(A)} \right\rceil.$$

By Theorem 2.1 we conclude that

$$d(G) = d(L_{\delta_G(A)}) = h_G(A).$$

By a result of Cossey, Gruenberg and Kovács [3, Theorem 3]

$$d(I_G) = \max\{h_G(M) \mid M \text{ an irreducible } G\text{-module}\}$$

thus, in particular, $d(I_G) \geq h_G(A) = d(G)$. Since $d(I_G) \leq d(G)$, we have an equality, hence $pr(G) = 0$. \square

3. EXAMPLES

In this section, we start by constructing a group that shows that the bound given in Theorem 1.2 cannot be improved. The same group provides an example of a coprimely invariably generated group which is not minimal-exponent. We then construct a family of examples which demonstrate that the property of coprime generation alone is not enough to constrain the minimal number of generators of a finite group.

Proposition 3.1. *Let $L = \text{ASL}_2(4) = \mathbb{F}_4^2 \rtimes \text{SL}_2(4) = V \rtimes \text{SL}_2(4)$, and let G be the crown-based power L_2 of L . Then G is coprimely invariably generated and $d(G) = 3$. Moreover, G has a proper subgroup with the same exponent.*

Proof. Note first that $|H^1(\text{SL}_2(4), V)| = 4$. Thus we may use Theorem 2.1 with $t = 2$, $\theta = 1$, $r_L(V) = 2$ and $s_L(V) = 1$ to see that $d(G) = 3$.

Let us now show that G is coprimely invariably generated. Let $z = (e_1, e_2) \in V \times V$, where e_1 and e_2 are linearly independent elements of V and note that, with this assumption on e_1 and e_2 , the normal closure $\langle z \rangle^G$ is the whole of V^2 . Choose $x \in \text{SL}_2(4)$ of order 3, and $y \in \text{SL}_2(4)$ of order 5. For any $a, b, c \in G$ and for $H = \langle x^a, y^b, z^c \rangle$, the quotient $HV^2/V^2 \cong \text{SL}_2(4)$, hence $\langle z^c \rangle^H = \langle z^c \rangle^G = \langle z \rangle^G = V^2$. Therefore $H = HV^2 = G$ and we conclude that G is invariably generated by x, y, z .

Finally, the subgroup $\{(l, l) \in L^2 \mid l \in L\}$ is a proper subgroup of G with the same exponent. \square

Proposition 3.2. *For any $t \in \mathbb{N}$ there exists a finite supersoluble group G such that G can be coprimely generated with $d(G) = t$ elements.*

Proof. Let $n = p_1 \cdots p_t$ be the product of the first t prime integers and let p be a prime such that n divides $p - 1$ (the prime p exists by Dirichlet's theorem). The cyclic group $C = C_n$ has a fixed point free multiplicative action on $V = \mathbb{F}_p$; set L to be the monolithic group $V \rtimes C$. Let G be the crown-based power L_t , then $d(G) = t + 1$ by Theorem 2.1.

Consider a generating set $\{x_1, \dots, x_{t+1}\}$ of C with $|x_i| = p_i$ if $i \leq t$ and $x_{t+1} = 1$. A well-known theorem of W. Gaschütz [12] states that if F is a free group with n generators, H is a group with n generators, and N is a finite normal subgroup of H , then every homomorphism of F onto H/N is induced by a homomorphism of F onto H . It follows that there exist w_1, \dots, w_{t+1} such that $G = \langle x_1 w_1, \dots, x_{t+1} w_{t+1} \rangle$. Clearly $|x_{t+1} w_{t+1}| = p$; on the other hand if $i \leq t$, then $C_{x_i}(V) = \{0\}$, and this implies that $|x_i w_i| = |x_i| = p_i$. \square

4. PROOF OF THEOREM 1.3

Theorem 4.1. *Let $L = A \rtimes H$ be a primitive monolithic group with abelian socle A and let $t \in \mathbb{N}$. If L_t is coprimely invariably generated, then*

$$t \leq \dim_{\text{End}_H(A)} A.$$

Proof. Let A be a p -group and set $G = L_t$. Assume that $\{g_1, \dots, g_u\}$ is a set of pairwise coprime elements that invariably generate G where g_i is a p' -element for every $i \neq 1$. Set $V = A^t$.

Note that, if $|g_i|$ is coprime to p and $g_i = vh$ where $v \in V$ and $h \in H$, then g_i is conjugate to an element of $\langle h \rangle$, since $\langle h \rangle$ is a Hall p' -subgroup of $V \langle h \rangle$; in particular g_i is conjugate to an element of H . Therefore, as $\{g_1, g_2, \dots, g_u\}$ invariably generates G , by taking suitable conjugates of g_2, \dots, g_u , we can assume that $g_2, \dots, g_u \in H$.

Consider $g_1 = vh$, where $v \in V$ and $h \in H$, and set $K = \langle h, g_2, \dots, g_u \rangle$. Since $KV = G = HV$ and $K \leq H$, we deduce that $K = H = \langle h, g_2, \dots, g_u \rangle$. Therefore,

$$G = \langle vh, g_2, \dots, g_u \rangle \leq \langle v, h, g_2, \dots, g_u \rangle \leq \langle v \rangle^H H$$

hence $G = \langle v \rangle^H H$ and $\langle v \rangle^H = V$, that is, v is a cyclic generator for the $\mathbb{F}_p H$ -module $V = A^t$. Let $v = (v_1, \dots, v_t)$. Switching to additive notation, the fact that v is a cyclic generator for the $\mathbb{F}_p H$ -module V implies that the elements v_1, v_2, \dots, v_t are linearly independent elements of the $\text{End}_H(A)$ -vector space A . In particular $t \leq \dim_{\text{End}_H(A)} A$, as required. \square

Proof of Theorem 1.3. Let G be a soluble, coprimely invariably generated group. Let L_t be a crown based power such that L_t is a homomorphic image of G and $d(G) = d(L_t) > d(L/A)$. Then L_t is coprimely invariably generated and L has abelian socle. Let $r_L(A)$ and $s_L(A)$ be as in Theorem 2.1.

Since L is soluble, we see from Theorem 2.3 that $s_L(A) = 0$. Moreover Theorem 4.1 implies that $t \leq r_L(A)$, and thus $\lceil (t + s_L(A))/r_L(A) \rceil = 1$. As

$d(L_t) > d(L/A)$, by Theorem 2.1 we conclude that

$$d(L_t) = \theta + \left\lceil \frac{t + s_L(A)}{r_L(A)} \right\rceil \leq 2,$$

as required. \square

5. PROOF OF THEOREMS 1.2 AND 1.4

Let L be a finite monolithic group whose socle A is non-abelian and let π be a set of primes. For every $l \in L$, define a_l to be the number of A -conjugacy classes of π -elements L which are contained in lA . Then set

$$a_\pi = \max\{a_l \mid l \in L\}.$$

As usual, for an integer n , the set of prime divisors of n is denoted $\pi(n)$.

Theorem 5.1. *Let L be a finite monolithic group whose socle A is non-abelian and let t be a positive integer. If the set $\{g_1, \dots, g_u\}$ invariably generates L_t , then $t \leq \prod_i a_{\pi(|g_i|)}$.*

Proof. Assume that $\{g_1, \dots, g_u\}$ invariably generates L_t , and set $\pi(|g_i|) = \pi_i$, for every i . Note that, by the definition of L_t , $g_i = (x_{i1}, \dots, x_{it})$ where x_{i1}, \dots, x_{it} belong to the same coset $l_i A$ for some $l_i \in L$; in particular x_{i1}, \dots, x_{it} are π_i -elements of $l_i A$.

If there exist r and s such that $x_{is} = x_{ir}^y$ for some $y \in A$, then by replacing g_i by a suitable conjugate we can assume that $x_{is} = x_{ir}$ (more precisely, we take the conjugate of g_i by the element $\bar{y} = (1, \dots, y, \dots, 1) \in L_t$, where y is in the r -th position). Let $a = \prod_i a_{\pi_i}$. If $t > a$, then it follows from the definition of a_π that there exist $r, s \in \{1, \dots, t\}$ with $r \neq s$ such that $x_{ir} = x_{is}$ for every $i \in \{1, \dots, u\}$. But then $\langle g_1, \dots, g_u \rangle \leq \{(l_1, \dots, l_t) \in L_t \mid l_r = l_s\}$ which is a proper subgroup of L_t , a contradiction. \square

Lemma 5.2. *Let L be a monolithic primitive group with non-abelian socle A and let π be a set of primes. Then*

$$a_\pi \leq k_\pi(A).$$

Proof. Let l be a π -element of L such that $a_l = a_\pi$. Set $X = \langle l \rangle A$. Let $x \in lA$. Since $X/A = \langle xA \rangle$, we have $X = AC_X(x)$ whence every X -conjugacy class in lA is a single A -orbit. In particular a_l coincides with the number of X -conjugacy classes of π -elements in the coset lA .

By [11, Theorem 1.6], a_l is precisely the number of A -conjugacy classes of π -elements in A which are invariant under X , whence $a_l \leq k_\pi(A)$. \square

Lemma 5.3. *Let L be a monolithic primitive group with non-abelian socle $A = S^n$, and let π_1, \dots, π_u be disjoint sets of primes. Then*

$$\prod_{i=1}^u a_{\pi_i} \leq \frac{|A|}{2n|\text{Out } S|}.$$

Proof. By Lemma 5.2, we may bound $a_{\pi_i} \leq k_{\pi_i}(A)$ for all i . As $A = S^n$, we get $k_{\pi_i}(A) = k_{\pi_i}(S)^n$. Now consider a partition $\tilde{\pi}_1, \dots, \tilde{\pi}_u$ of $\pi(|S|)$ with the property that $\tilde{\pi}_i \supset \pi_i \cap \pi(|S|)$: clearly $k_{\pi_i}(S) \leq k_{\tilde{\pi}_i}(S)$. It follows from Theorem 1.7 (whose proof is in Section 7) that

$$\prod_{i=1}^u k_{\tilde{\pi}_i}(S) \leq \frac{|S|}{2|\text{Out } S|}.$$

Therefore

$$\prod_{i=1}^u a_{\pi_i} \leq \prod_{i=1}^u k_{\pi_i}(A) = \prod_{i=1}^u k_{\pi_i}(S)^n \leq \prod_{i=1}^u k_{\tilde{\pi}_i}(S)^n \leq \frac{|S|^n}{2^n |\text{Out } S|^n} \leq \frac{|A|}{2n |\text{Out } S|}$$

as required. \square

Lemma 5.4. *Let L be a monolithic primitive group with non-abelian socle $A = S^n$. If L_t is minimally d -generated (i.e. $d(L_t/N) < d(L_t) = d$ for every $1 \neq N \triangleleft L_t$) and*

$$t \leq \frac{|A|}{2n |\text{Out } S|}$$

then $d = 2$ (and $t = 1$).

Proof. Set $d_L = d(L)$ and note that $d_L \geq 2$ since L has non-abelian socle. Let X be the subgroup of $\text{Aut } S$ induced by the conjugation action of $N_G(S_1)$ on the first factor S_1 of $A = S_1 \times \dots \times S_n$, with $S \cong S_i$ for each $1 \leq i \leq n$. As in the proof of Lemma 1 in [5],

$$|C_{\text{Aut } A}(L/A)| \leq n|S|^{n-1}|C_{\text{Aut } S}(X/S)|$$

and therefore

$$|C_{\text{Aut } A}(L/A)| \leq n|S|^{n-1}|\text{Aut } S| = n|A||\text{Out } S|.$$

By Theorem 2.5, $P_{L,A}(d_L) \geq 1/2$. So the assumptions give that

$$t \leq \frac{1}{2} \frac{|A|}{n|\text{Out } S|} \leq \frac{P_{L,A}(d_L)|A|^2}{n|A||\text{Out } S|} \leq \frac{P_{L,A}(d_L)|A|^{d_L}}{|C_{\text{Aut } A}(L/A)|}.$$

By Theorem 2.4 this implies that $d = d(L_t) = d_L$. As L_t is minimally d -generated, it follows that $t = 1$; in particular, L is minimally d -generated. Now, by the main theorem in [17], $d(L) = \max\{2, d(L/A)\}$, and again by minimality, we conclude that $d = d(L) = 2$. \square

Proof of Theorem 1.2. Let G be a coprimely invariably generated group and let $d = d(G)$. As remarked in Section 2, there exists a monolithic primitive group L with socle A and an integer t , such that L_t is a quotient of G and $d = d(L_t) > d(L/A)$. Moreover L_t is coprimely invariably generated.

If A is abelian, then we can apply Theorem 2.1: since $d(L_t) > d(L/A)$ and, by Theorems 2.2 and 4.1, $s_L(A) < r_L(A)$ and $t \leq r_L(A)$, it follows that

$$d(G) = d(L_t) = \theta + \left\lceil \frac{t + s_L(A)}{r_L(A)} \right\rceil \leq \theta + 2 \leq 3.$$

If A is non-abelian and $\{g_1, \dots, g_u\}$ are coprime invariable generators of L_t , then by Theorem 5.1, $t \leq \prod_{i=1}^u a_{\pi(|g_i|)}$. Then by Lemma 5.3

$$\prod_{i=1}^u a_{\pi(|g_i|)} \leq \frac{|A|}{2n|\text{Out } S|}.$$

Thus

$$t \leq \frac{|A|}{2n|\text{Out } S|}$$

and by Lemma 5.4 we conclude that $d(G) = d(L_t) = 2$. \square

Proof of Theorem 1.4. Let G be a coprimely invariably generated group. Assume, by way of contradiction, that $pr(G) > 0$. Let L and $t \in \mathbb{N}$ be such that L is a monolithic primitive group with socle A and L_t is a homomorphic image of G , with $d(L_t) = d(G) = d$ and $d > d(L/N)$.

If A is abelian, then $pr(G) = 0$ by Theorem 2.6, a contradiction. If A is non-abelian, then arguing as in the proof of the non-abelian case of Theorem 1.2, we conclude that $d = d(L_t) = 2$. Thus again $pr(G) = 0$. \square

6. PROOF OF THEOREM 1.6

In the following proof, by $[a, b]$ we denote the lowest common multiple of integers a and b .

Proof of Theorem 1.6. We make use of the invariable generators given in [14], where it is proved that every finite simple group is invariably generated by two elements. For the classical groups, the orders given in [14] are for the quasisimple groups, so we must adjust their values to get coprime projective orders.

For the alternating groups, the generators given in [14, Proof of Lemma 5.2] are of coprime orders.

For the special linear groups in dimension $n \geq 3$, the invariable generators in [14] have orders $(q^n - 1)/((q - 1, n)(q - 1))$ and $(q^{n-1} - 1)/(q - 1)$, which are coprime. The given generators for $\text{PSL}_2(q)$ are also always of coprime order. For the unitary groups and the orthogonal groups other than $\text{P}\Omega_{4k+2}^-(q)$ with q odd and $\Omega_8^+(q)$ with $q \leq 3$, the given generators are coprime. For $\text{P}\Omega_{4k+2}^-(q)$ it suffices to take the square of the second generator in [14] to produce coprime invariable generators.

For the symplectic groups in dimension $2m \geq 4$, the given generators have orders $(q^m + 1)/(q - 1, 2)$ and $[q^{m-1} + 1, q + 1]$, so when m is even as the corresponding elements of the simple group are coprime. When m is odd we choose three elements: one of order $(q^m + 1)/(2, q - 1)$, one of order $(q^{m-1} + 1)/2$, and one of order $(q^m - 1)/2$. These are coprime, and it follows from [19, Theorem 1.1] that these elements invariably generate $\text{PSp}_{2m}(q)$.

Of the classical groups, this leaves only $\Omega_8^+(2)$ and $\text{P}\Omega_8^+(3)$. For $\Omega_8^+(2)$, choose an element a from class 5A, an element b from class 7A and an element c from class 9A. The only maximal subgroup of $\Omega_8^+(2)$ to contain elements

of all three of these orders is $S_6(2)$, and $S_6(2)$ has no maximal subgroups that contain elements of all three of these orders, so if $\langle a^x, b^y, c^z \rangle = H \neq \Omega_8^+(2)$ then $H \cong S_6(2)$, and H contains 5A elements of $\Omega_8^+(2)$. The outer automorphism group of $\Omega_8^+(2)$ acts on the three classes of $S_6(2)$ in the same way as it acts on the three classes of elements of order 5. Thus specifying that H contains 5A elements tells us which $\Omega_8^+(2)$ -conjugacy class of groups $S_6(2)$ we have, and in particular without loss of generality $H = \langle a, b, c \rangle$. Such a group contains no elements from (a fixed) one of classes 2C or 2D, so we let d be an element from this class. Then a, b, c and d coprimely invariably generate $\Omega_8^+(2)$.

For $G = P\Omega_8^+(3)$, we first note that G contains three classes of elements of order 5, two of order 13 (one of which contains powers of the other), and only one of elements of order 7. So we let $a \in 7A$, $b \in 13A$ and $c \in 5A$. Order considerations show that the only possible maximal subgroup to contain $\langle a^x, b^y, c^z \rangle$ is $\Omega_7(3)$, and that given a^x, b^y, c^z , they are contained in at most one copy of $\Omega_7(3)$. There are six classes of groups $\Omega_7(3)$ in G , with stabiliser D_8 , and these classes are cycled in two 3-cycles by the triality automorphism, which also permutes the three classes of elements of order 5 in G . Thus there are most two G -conjugacy classes of groups $\Omega_7(3)$ that are generated by $\langle a^x, b^y, c^z \rangle$ as x, y, z vary. Now, $\Omega_7(3)$ contains four conjugacy elements of order 9, which form three orbits under $\text{Aut } \Omega_7(3)$. Conversely, $\Omega_8^+(3)$ contains 14 conjugacy classes of elements of order 9, forming orbits of length 6, 4 and 4. Consider the orbit of length 6. The two G -conjugacy classes of groups $\Omega_7(3)$ intersect at most four of these classes, so let d be an element of order 9 in one of the remaining two classes. Then a, b, c, d coprimely invariably generate G .

For all of the exceptional groups except $E_7(q)$, the invariable generators given in [14] are coprime. Thus we need only consider $E_7(q)$. By [13, Table 6], elements of order $(q+1)(q^6 - q^3 + 1)/(2, q-1)$ are contained only in a copy of ${}^2E_6(q)_{sc}.D_{q+1}$. Since the order of $E_7(q)$ is divisible by $q^{14} - 1$, we may find an element of order a Zsigmondy prime for $q^{14} - 1$ in $E_7(q)$. Such a prime does not divide the order of ${}^2E_6(q)$ or $q+1$, so gives a pair of invariable generators for $E_7(q)$.

For the sporadics and the Tits group, [13, Table 9] lists carefully chosen conjugacy classes of elements of the sporadics groups, together with a complete list of the maximal subgroups containing those conjugacy classes. It suffices to check that in each case there exists an element of order coprime to the given one that lies in none of the listed maximal subgroups. \square

7. PROOF OF THEOREM 1.7

In this section we prove Theorem 1.7. First, we need a preliminary lemma.

Lemma 7.1. *Assume that G is a finite group and let $\pi \subseteq \pi(G)$. Then $k_\pi(G) \leq |G|_\pi$. In particular if $\pi = \{p\} \cup \tilde{\pi}$, then $k_\pi(G) \leq k_p(G) \cdot |G|_{\tilde{\pi}}$.*

Proof. We prove that $k_\pi(G) \leq |G|_\pi$ by induction on $|\pi|$. The case $|\pi| = 1$ is an immediate consequence of the Sylow Theorems. Assume $\pi = \{p\} \cup \tilde{\pi}$. Let g be a π -element of G ; we may write $g = ab$ where a is a p -element and b is a $\tilde{\pi}$ -element and both are powers of g . Up to conjugacy we have at most $k_p(G)$ choices for a . For a fixed choice of a we have to count the number of b . Notice that $b \in H = C_G(a)$. Moreover if b_1 and b_2 are conjugate in H then ab_1 and ab_2 are conjugate in G . Hence the number of choices of b is bounded by the number of conjugacy classes of $\tilde{\pi}$ elements in H , and by induction this number is at most $|H|_{\tilde{\pi}} \leq |G|_{\tilde{\pi}}$. Thus $k_\pi(G) \leq |G|_\pi$ as required.

By the same argument, we now have that

$$k_\pi(G) \leq k_p(G)k_{\tilde{\pi}}(H) \leq k_p(G)|H|_{\tilde{\pi}} \leq k_p(G)|G|_{\tilde{\pi}}.$$

□

We in fact prove a slightly stronger version of Theorem 1.7, which we state now. Let $\mathcal{S} = \{A_n : n \leq 7\} \cup \{L_2(q) : q \in \{7, 8, 11, 27\}\} \cup \{L_3(4)\}$.

Theorem 7.2. *Let S be a finite simple group and let π_1, \dots, π_u be a partition of $\pi(S)$. Then*

$$\prod_{i=1}^u k_{\pi_i}(S) \leq \frac{|S|}{2|\text{Out } S|}.$$

Furthermore, if $S \notin \mathcal{S}$, then there exists a prime p dividing $|S|$ such that

$$k_p(S) \leq \frac{|S|_p}{2|\text{Out } S|}.$$

Proof. For groups in \mathcal{S} , this is a direct calculation using their conjugacy classes. For the remaining groups, the first claim follows from the second and Lemma 7.1. The alternating case is considered in Lemma 7.3, below. The linear and unitary groups and the symplectic and orthogonal groups are dealt with in Lemmas 7.4 and 7.5, respectively. The exceptional case is completed in Lemma 7.6. For the sporadics, this is a straightforward exercise, using [2]. □

Lemma 7.3. *Let $S = A_n$ for some $n \geq 7$. Then there exists a prime r dividing $|S|$ such that S has at most one conjugacy class of nontrivial r -elements. Furthermore, if $n \geq 8$ then there exists a prime p dividing $|S|$ such that*

$$k_p(S) \leq \frac{|S|_p}{2|\text{Out } S|}.$$

Proof. First let $k = \lfloor n/2 \rfloor$. Then Bertrand's postulate states that for $k \geq 4$, there exists a prime r such that $k \leq n/2 < r < 2k - 2 \in \{n-2, n-3\}$, so the first claim follows (after verifying that $r = 5$ works when $n = 7$).

As for the second claim, note that $|\text{Out } S| = 2$. For $n = 8$, we use $k_2(S) = 5$ whilst $|S|_2 = 2^6$. For $n = 9$, we use $k_3(S) = 6$ whilst $|S|_3 = 3^4$. For $n \in \{10, 11, 12, 13\}$ we use $k_5(S) = 3$. We may therefore assume that

$n \geq 14$ and $n - 2 > p = r \geq 11$. Thus $k_p(S) = 2$, whilst $\frac{|S|_p}{2|\text{Out } S|} \geq 11/4 > 2$, so the result follows. \square

Lemma 7.4. *Let $S \cong L_n(p^e), U_n(p^e)$ be simple, and assume that $S \notin \{L_2(q) : q \in \{4, 5, 7, 8, 9, 11, 27\}\} \cup \{L_3(4)\}$. Then*

$$k_p(S) \leq \frac{|S|_p}{2|\text{Out } S|}.$$

Proof. By [15, Lemma 1.4], $k_p(S) \leq np(n) + 1$, where $p(n)$ is the partition function of n . Since $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$, where the sum is over the pentagonal numbers less than n and the sign of the k th term is $(-1)^{\lfloor (k-1)/2 \rfloor}$, we may bound $np(n) + 1 \leq n2^n$.

First suppose that $n = 2$, so that $|S|_p = q$. Then without loss of generality $S \cong L_2(p^e)$. Here $k_p(S) = 2$ for $p = 2$, and 3 for p odd, whilst $|\text{Out } S|$ is e for $p = 2$ and $2e$ for p odd. Thus for $p = 2$ we must check that $2^e \geq 2 \cdot 2e$, which holds for all $e \geq 4$. For p odd we require $p^e \geq 12e$, which clearly holds for all e when $p \geq 13$. If $p = 3$ this yields $e \geq 4$, and when $5 \leq p \leq 11$ this yields $e \geq 2$.

Next suppose that $n = 3$, so that $|S|_p = q^3$. Suppose first that $S \cong L_3(p^e)$. If $q \equiv 1 \pmod 3$ then $k_p(S) = 5$ and $|\text{Out } S| = 6e$, so we require $p^{3e} \geq 60e$, which holds for all such $q > 4$. If $q \equiv 0, 2 \pmod 3$ then $k_p(S) = 3$ and $|\text{Out } S| = 2e$, so we require $p^{3e} \geq 12e$, which holds for all $q > 2$ (but recall that $S \not\cong L_3(2) \cong L_2(7)$). Suppose next that $S \cong U_3(p^e)$. In this case, if $q \equiv 2 \pmod 3$ then $k_p(S) = 5$, whilst if $q \equiv 0, 1 \pmod 3$ then $k_p(S) = 3$. Since $U_3(2)$ is not simple, and $|\text{Out } S| = (3, q+1) \cdot 2e$, the result follows by a similar calculation to that for $L_3(q)$.

We now consider the general case. We bound k_p by $np(n) + 1 \leq n2^n$, whilst the order of a Sylow p -subgroup of S is $q^{n(n-1)/2}$ and

$$|\text{Out } S| \leq 2(q-1) \log_p q < q^2.$$

If $n2^n \geq q^{n^2/2-n/2-2}/2$ then $(n, q) \in \{(4, 2), (4, 3), (5, 2)\}$. In fact $k_2(L_4(2)) = 5 < 2^6/4$, whilst $k_3(L_4(3)) = 7 < 3^6/8$ and $k_2(L_5(2)) = 7 < 2^{10}/4$, so the result follows. \square

Lemma 7.5. *Let S be a simple symplectic or orthogonal group, of rank n over \mathbb{F}_{p^e} . Then*

$$k_p(S) \leq \frac{|S|_p}{2|\text{Out } S|}.$$

Proof. Here $|S|_p \geq q^{n^2-n}$ and $|\text{Out } S| \leq 2(q-1, 2)^2 \log_p q$, which is less than q^2 for all q . By [15, Lemmas 1.4 and 1.5] if S is symplectic then

$$k_p(S) \leq p(2n)2^{(2n)^{1/2}} < 6^n$$

(where $p(n)$ is the partition function of n), whilst if S is orthogonal then

$$k_p(S) \leq 2(n, 2)p(2n+1)2^{(2n+1)^{1/2}} < 6^n.$$

If $n = 2$ then $q > 2$ and S is symplectic, so that $|S|_p = q^4$ and $|\text{Out } S| = 2(q, 2) \log_p q$, whilst $k_p(S) \leq 7$, so the result follows for all q .

If $n = 3$ then $k_p(S) \leq 60,187$ for S symplectic or orthogonal, respectively, so the result is immediate for $q \geq 5$, and for the remaining q we check that in fact $k_p(S) \leq 16$.

If $n = 4$ then $k_p(S) \leq 156$ for S symplectic and 960 for S orthogonal, so the result is immediate for $q \geq 7$. For $2 \leq q \leq 5$ we verify that in fact $k_p(S) \leq 81$, which completes the proof.

If $n \geq 5$ the result follows immediately from the 6^n bounds, for all q . \square

Lemma 7.6. *Let $S \cong {}^r X_l(p^e)$ be a simple group of exceptional type. Then*

$$k_p(S) \leq \frac{|S|_p}{2|\text{Out } S|}.$$

Proof. We use the results cited in [15, Proof of Lemma 1.5] to bound $k_p(S)$ for each family. Let $q = p^e$.

If $S \cong F_4(q), E_6(q), {}^2E_6(q), E_7(q), E_8(q)$, then $|S|_p \geq q^{24}$ and $|\text{Out } S| \leq 6 \log_p q < q^3$, whilst $k_p(S) \leq 202$ so the result is clear.

If $S \cong G_2(q)$ then $|S|_p = q^6$ and $|\text{Out } S| \leq 2 \log_p q < q$, whilst $k_p(S) \leq 9$. If $S \cong {}^2B_2(q)$ then $|S|_p = q^2$ and $|\text{Out } S| = \log_p q$, whilst $k_p(S) = 4$, so the result holds for all $q > 2$, however ${}^2B_2(2)$ is not simple. If $S \cong {}^2D_4(q)$ then $|S|_p = q^{12}$ and $|\text{Out } S| = 3 \log_p q < q^2$, whilst $k_p(S) \leq 8$, so the result is clear. If $S \cong {}^2G_2(q)$ then $q \geq 27$ with $|S|_p = q^3$ and $|\text{Out } S| = \log_p q$, whilst $k_p(S) \leq 10$, so the result holds for all q . Finally, if $S \cong {}^2F_4(q)$ then $|S|_p = q^{12}$ and $|\text{Out } S| = \log_p q$, whilst $k_p(S) < 35$. \square

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